

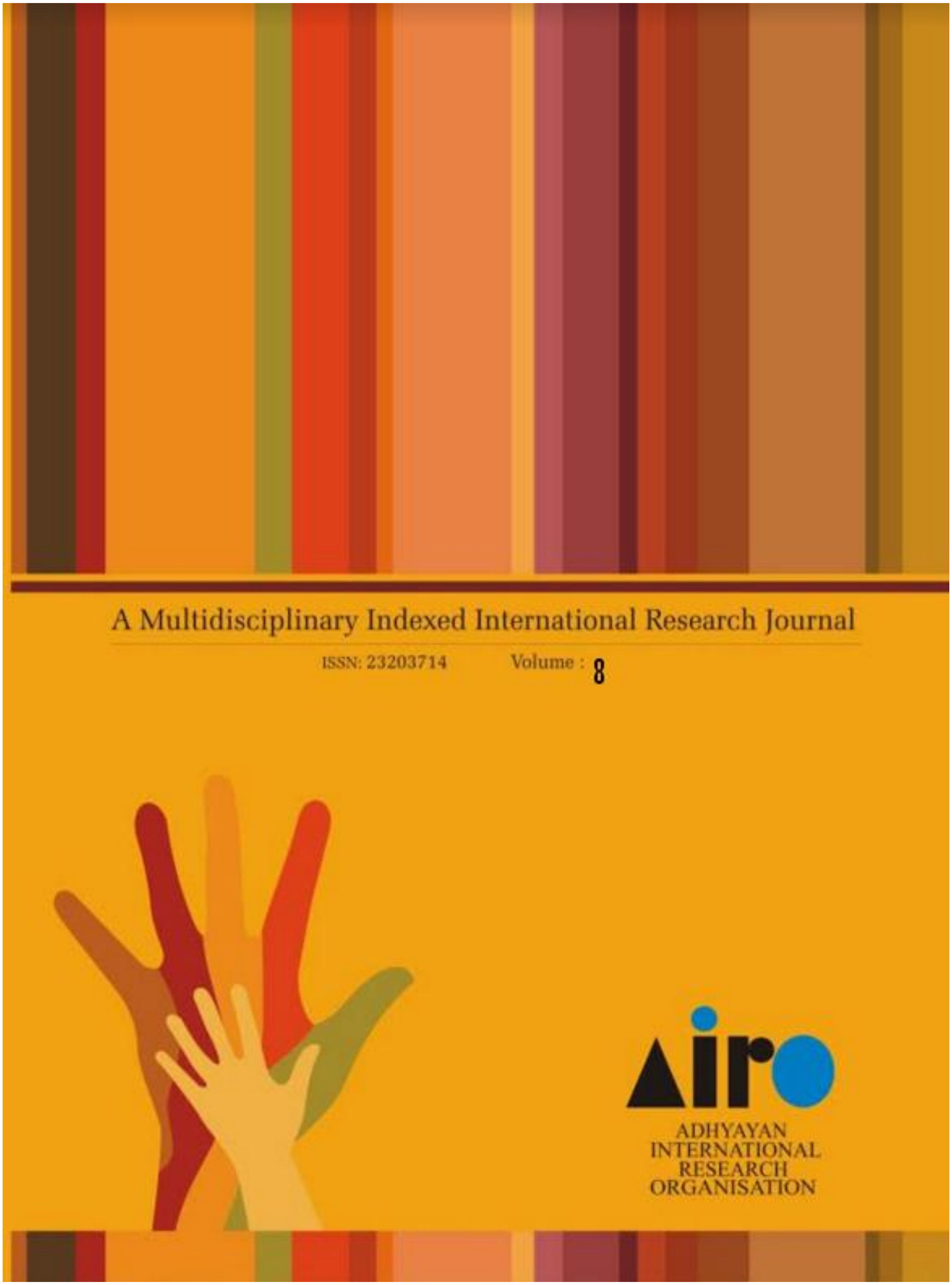
A Multidisciplinary Indexed International Research Journal
ISSN: 23203714
Volume : 8
ADHYAYAN
INTERNATIONAL
RESEARCH
ORGANISATION

Fractional Differential Problem Solution using Karasnoselskii Fixed Point Theorem

Harish Kumar, M.D.U Rohtak

Abstract: - Burton & Colleen Kirk continued Karasnoselskii's approach by adding a concept of Schauderon fixed points from a priori bounds with Banach's theorem. In this paper a basic understanding of Karasnoselskii fixed point theoremfor fractional differential problem & its solution is introduced.This fixed point theorem fundamentals involve two operators that justified by using Banach's contraction with Schauder's fixed point principle. This paper gives the solution to fractional differential problem .The study of connected systems involving fractional differential equations is also important as such this type of systems occur in various problems of applied nature. This work is focused to the study of the existence of solution to the system of fractional hybrid differential equations (Caputo fractional derivative & Reimann-Liouville fractional integral).

I. INTRODUCTION

It is illustrious that Krasnoselskii's fundamental theorem may linked with Banach as well as Schauder's fixed point theorems. If a compact operator has the fixed point functionality, under a small context, then this concepts can be inherited. The addition of two different operators is clearly seen in delay integral equations and neutral functional equations, which have been revealed broadly in [1]. Krasnoselskii justified in his work that the addition of K+Lhas a fixed point in M, if (a) K is continuous and compact operator, (b) $Kx + Ly \in M$ for every $x, y \in M$ and (c) L is a strict contraction operator. Therefore, in various applications, the proof of (b) is quite complex to demonstrate as well as assumption of (c) is also little bit restrictive.

Presently, as a conditional approach to short out these complexities, many genuine works have illustrated with variety of ways and directions of weakening conditions (b) & (c). In order to lift condition (c), Burton derived the fundamentals of a large contraction mapping and generalized this notorious result in a wide setting [2]. Burton and Colleen Kirk in 1998 [3] continued & improved Krasnoselskii's concept by merging a result of Schaefer [4] on fixed points from a priori bounds with Banach's theorem. Barroso [5] described the further enhancement for (b). If $\lambda \in (0, 1)$, $u = \lambda Lu + Kv$ for some $v \in M$, then $u \in M$. About (c), Most of the researchers achieved their objectives by directly or indirectly making the assumption that $I - B$ is continuously invertible. Actually, it is more exciting to unveil the case when $I - L$ is not injective. In this present work, we analyze this kind of generalization by looking for the multi-valued operator $(I - L)^{-1}K$ achieving a fixed point in M.

Nonlinear differential equations (NDE) are typical tools in the modeling of nonlinear actual experience matching to a great collection of events, in relation with various domains of the physical sciences as well as technology. For certain, they occur in the review of the air motion, in fluids dynamics, electrostatic, electromagnetism as well as the supervision of nonlinear procedures, between others [6]. The resolution of NDE needs, in common, the evolution of various methods in order to figure out the existing & other required properties of the solutions. Moreover, there are still many open difficulties analogues to the solvability of nonlinear systems, rather than the matter that this is a domain where researches are keep on continuing. Perturbation methods are powerful in the nonlinear studyto analyze the dynamical systems derived by nonlinear differential (Caputo fractional derivative) and integral equations (Reimann-Liouville fractional integral). It is clear that, few differential equations denoting a certain dynamical system which has no analytical solution, thus the perturbation of these problems can be beneficial. The perturbed differential equations are classified into different types. A valuable class of these such perturbations is represented by a hybrid differential equation (quadratic perturbation of a nonlinear differential equation) [7].

Latterly, the fractional differential equations (hybrid) have been drawing more attention [8], & many researchers proposed the theory about hybrid differential equations. Rather than this, hybrid fixed point fundamentals can also be used to construct the existing concepts for the hybrid equations [9]. Dhage as well as Jadhav [10] described the Ist-order hybrid differential equation with linear perturbations of second type is given below:

$$\left\{\begin{array}{l} \frac{d}{dt}[x(t)-f(t,x(t))]=h(t,x(t)), \\ x(t_0)=x_0\in\Re \end{array}\right\}, \text{ such that } t\in J$$

Where, J=[t_0,t_0+a)]in □ for some constant value t_0,a∈□, witha>0, and *f*,*h*∈C(J×□,□). They justified the occurrence of the maximum as well as minimal solution for this equation. Moreover, they estimated some fundamentals outcomes regarding the strict as well as non-strict differential inequalities.Undoubtedly, the fractional differential equations have beenprominently used in modeling various physical concepts and have been stated through researchers in couple of years [11]-[15], thus it appear to justify a distinct work of their fundamentalsconcurrently to the basics of ordinary differential equations. Yeng & Teng in 2013 derived this problem w.r.t the following fractional hybrid differential equation involving the Reimann-Liouvilledifferential operators of order $0<\gamma<1$, with linear perturbations of second type:

$$\left\{\begin{array}{l} D^{\gamma}[x(t)-f(t,x(t))]=h(t,x(t)), \\ x(t_0)=x_0\in\Re \end{array}\right\}, \text{ such that } t\in J$$

f,*h*∈*C*(J×□,□). Next to that, using mixed Lipchitz& Caratheodory theorems admitted them to prove basics for fractional hybrid differential equations. The functional description of coupled systems consisting fractional differential equations is playing a vital role in distinct problems of applied nature[16]-[19]. Later in 2009, Su [20] illustrated a two-point boundary value problem for a coupled system of fractional differential equations also one more researcher [21] examined the fundamental outcomes of coupled nonlinear fractional reaction-diffusion equations.

In the above shown literature works, our main objective of this paper is to proof the existence of solution to the following system of fractional hybrid differential equations of order 0<γ<1:

$$\left\{\begin{array}{l} D^{\gamma}[x(t)-f(t,x(t))]=h(t,y(t),I^{\Phi}(y(t))) \\ D^{\gamma}[y(t)-f(t,y(t))]=h(t,x(t),I^{\Phi}(x(t))) \\ x(0)=0, y(0)=0 \end{array}\right\}$$ Such that t ϵ J, 0<γ<1, Φ>0

(1)

II. BASIC FUNDAMENTAL CONCEPTS

Let's assume *C*(J×□×□,□) denote the family of continuous functions *f*:J×□×□ →□ andlet's assume *C*(J×□×□,□) denote the family of functions h:J×□×□ →□ such that

a) the map t→*h*(t,x,y) is measurable for each x,y∈□,
b) the map x→*h*(t,x,y) is continuous for each x∈□,
c) the map y→*h*(t,x,y) is continuous for each y∈□,

The family *C*(J×□×□,□) is known as the Caratheodory family of functions on J× □× □,which are Lebesgue integrable when it is bounded by a Lebesgue integrable function over J.

We required few identicaldefinitions of the basic fundamentals. The following is a discussion of few of the fundamentals we will require.

Definition 1 [22]:
The mathematical form of the Riemann-Liouville (RL) fractional integral operator of order Φ>0, of function *f*∈ Ѣ ($□^{+}$) is defined as

$$I^{\Phi} f(t)=\frac{1}{\Gamma(\Phi)}\int_{0}^{t}(t-z)^{\Phi-1} f(z)dz$$

Definition 2 [22]:
Let's assume Φbe the positive real no., such that m−1<Φ≤m, m∈ Ň and $f^{(m)}(x)$ exists, a function of family *C*. Then Caputo fractional derivative of *f* is denoted as:

$${}^{c}D^{\Phi} f(x)=\frac{1}{\Gamma(m-\Phi)}\int_{0}^{t}(t-z)^{m-\Phi-1} f^{(m)}(z)dz$$

Definition 3 [22]:
The Riemann-Liouville fractional derivative of order Φ>0, of a continuous function *f*:(0,∞)→Ris described as

$$D^{\Phi} f(t)=\frac{1}{\Gamma(n-\Phi)}(\frac{d}{dt})^{n}\int_{0}^{t}(t-z)^{n-\Phi-1} f^{(m)}(z)dz$$ Where, n=[Φ]+ 1.

Lemma 1 [22]-[23]:
Let's assume 0<Φ<1, and function *f*∈ Ѣ¹(0, 1) then,
(E1): The equality $D^{\Phi} I^{\Phi} f(t) = f(t)$ holds.
(E2): The equality $D^{\Phi} I^{\Phi} f(t) = f(t) - \frac{[D^{\Phi-1} f(t)]_{t=0}}{\Gamma(\Phi)} t^{\Phi-1}$ holds everywhere on *J*.

Now the fixed point theorem introduced by Burton in Banach spaces [2]-[3].

Lemma 2 [2]-[3]:
Let's assume U be a non-empty, closed, convex, and bounded subset of a Banach space S and let's assume A: S→S and B: U→S be two operators such that

a) A is a contraction with constant Φ<1,
b) B is completely continuous,
c) Kx + Ly =>x ∈ U for every x, y ∈ U

Then the operator equation Kx+Lx=x has a solution in U.
Now, we reveal the concepts of a coupled fixed point for a bivariate mapping.
Definition 4 [24]:
An element (a, b) ∈ A×A is called a coupled fixed point of a mapping M: A×A→A
If M(a, b) = a and M(b, a) = b.
Let's denote by □ the family of all functions□ :□$^+$→ □$^+$ fulfilling □ (r)<rfor r>0 and □ (0) =0
By a solution of the fractional hybrid differential equations system (1) we mean a function (a, b)∈*AC* (*J*× □× □) such that:

- the function t→ a - f(t, a) is absolutely continuous for each a ∈ □ and
- (a, b)satisfies the system of (FHD) equations in (1).

Where, *AC* (*J*× □× □) is the space of absolutely continuous real-valued functions characterized on *J*.

III. RESULTS

Throughout this result section, let's assume A= C (J× □)equipped with the chebyshev norm. Obviously, it is a Banach space with respect to point-wise operations and the chebyshev norm.
Define scalar multiplication and a sum on A×A as follows:
(K1, L1)+(K2, L2)=(K1+K2, L1+L2)and ρ*(K, L)=(ρK, ρL), for ρ ∈ □ .Then A×A is a vector space on □ .
In the following lemma 3 we illustrate a certain Banach space which is used in our findings.

Lemma 3:
Let's assumeÃ: - A×A.
Define: $\|(K,L)\| = \|K\| + \|L\|$, then Ã is a Banach space for above described norm.
Proof: Clearly Ã is Banach space and ||.|| is a norm on Ã.
Therefore, we prove a couple of fixed point theorems which is generalization of Burton [2]-[3].

Theorem 1:
Let's assume Ũ be a non-empty, closed, convex, and bounded subset of a Banach space Ṩ and let's assume A: Ṩ →Ṩ and B: Ũ→Ṩ be two operators such that

a) there exists φ_A∈Φsuch that for all x,y∈Ṩ , we have

‖Ax−Ay‖≤σ*φ_A*(‖x−y‖),for some constant σ>0,

b) B is completely continuous,
c) x =Ax+By⇒x∈Ũ, for all y∈Ũ.

Then the operator T(x,y)=Ax+By has at least a coupled fixed point in Ũ whenever σ<1.

Proof: It is easy to verify that Ũ is a non-empty, closed, convex, and bounded subset of a Banach space Ṩ . Define A: Ṩ → Ṩ and B: Ũ → Ṩ by

A(x, y)= (Ax, Ay), B(x, y)= (By,Bx).
It is enough to prove A(x, y)+B(x, y)=(x, y) has minimum one solution just because of
[T(x, y), T(y, x)]= {Ax+By, Ay+Bx}
= {Ax, Ay+ By, Bx}
= {A(x, y)+ B(x, y)}=(x, y),
Which shows that T(x, y) has at least one coupled fixed point. We demand that the operators A, B satisfy all the conditions presented in lemma 2 on the Banach space Ṩ .
First, we demonstrate that A is Lipschitzian. With the help of a (Theorem1), for every x=(x1,x2),y=(y1,y2)∈Ṩ we have
||Ax - Ay||=||(Ax1,Ax2)-(Ay1, Ay2)||
=|| (Ax1- Ay1, Ax2-Ay2)||
=||Ax1-Ay1||+ ||Ax2-Ay2||
≤σ (φ(||x1-y1||+φ(||x2-y2||)
<σ [(||x1-y1||+(||x2-y2||)]
= σ||(x1-y1, x2-y2)||
= σ ||x-y||
Which indicates that A is a contraction with constant σ. Further, we demonstrate that B is a compact and continuous operator on Ũ.

Let's assume $(p_n)= (p_{1n}, p_{2n})$be a sequence in Ũ converging to a point p=(p1,p2)∈Ũsince B is continuous we have the following outcomes

$$\lim_{n\to\infty} Bp_n = (\lim_{n\to\infty} Bp_{2n}, \lim_{n\to\infty} Bp_{1n})$$
$$= (Bp_{2n}, Bp_{1n}) = B(p_1, p_2) = Bp$$
, So it could be say that B is continuous.

Let's assume p= (p_1, p_2)∈Ũ, we have the followings
$||B(p_1, p_2)|| = ||(Bp_1,Bp_2))$
$= (||Bp_1|| + ||Bp_1||)$
≤ 2||BŨ||, for all p ∈Ũ.
Where, ||BŨ||= Chebyshev{||Bp||: p ∈Ũ}.
This illustrates that B is uniformly bounded on Ũ.

Let's assume ς >0, since BŨ is an equi-continuous set in Ṩ , hence there exists β>0
Such that t1,t2∈J, |t1−t2|<β implies that |Bp(t1)−Bp(t2)|≤ς for all p ∈Ũ.
Then for any p=(p1,p2)∈Ũ, we have

$$\left|Bp(t1) - Bp(t2)\right| = \left|(Bp_2(t1), Bp_1(t1)) - (Bp_2(t2), Bp_1(t2))\right|$$
$$= \left|(Bp_2(t1) - Bp_2(t2)), (Bp_2(t1) - Bp_1(t1))\right|$$
$$= \sqrt{(Bp_1(t1) - Bp_1(t2))^2 + (Bp_2(t1) - Bp_2(t2))^2} \le \sqrt{2}\varsigma$$

So B(Ũ) is an equi-continuous set in Ṩ . Thus B(Ũ) is compact by the concept (Arzelà-Ascoli theorem). As a result, B is a continuous and compact operator onŨ. So, B is completely continuous onŨ.
Further, we demonstrates that hypothesis (c) of Lemma 2 is satisfied. Let p =(p1,p2)∈Ṩ , q=(q1,q2)∈Ũsuch that p=Kp+Lq. Then by assumption (c, theorem 1), we have (p1,p2)=A(p1,p2)+B(q1,q2)

= (Ap1,Ap2)+(Bq2,Bq1)
=(Ap1+Bq2,Ap2+Bq1),
which implies that
p1=Ap1+Bq2,
p2=Ap2+Bq1.
So, by assumption (C, theorem 1), we have p1,p2∈Ũ. Thus, p ∈Ũ. Then all conditions of Lemma are justified and hence the operator equation Kx+Lx=x has at least one solution on Ũ. Thus, T(p,q)has at least one coupled fixed point and the proof is done.
Now by having Theorem 1, we analysis the existence of solution for the fractional hybrid differential equations system (1) under the assumptions described below.

(FH0) The function p→p−f(t,p) is increasing in □ for all t∈J.
(FH1)There exists a constant M≥L>0such that

$$\left| f(t,p(t)) - f(t,q(t)) \right| \le \frac{L(\left|p(t)-q(t)\right|)}{2(M+\left|p(t)-q(t)\right|)}$$, for all t∈J and p, q ∈□.

(FH2) Fix $F_0 = \max_{t\varepsilon J} | f(t,0) |$
(FH3) There exists a continuous function □ ∈*C(J, □)* such that
g{t,p(t),q(t)}≤□(t), p,q ∈□, t ∈ J
With the continuation of Lemma 1 we have the following lemma which is important in the existence outcomes.

Lemma 4 [16]:
Let's assume that hypothesis (FH0) holds, p ∈*C(J, □)* 0<Φ<1,
α>0, and *f*∈C(J×R,R)with *f*(0,0)=0. Then the unique solution of the initial value problem

Problem→ $$\begin{cases} D^{\Phi}[p(t) - f(q,q(t))] = q(t) \\ p(0) = 0 \end{cases}$$, t ∈ J
(2)

Solution is→ $$p(t) = f(t,p(t)) + \frac{1}{\Gamma(\Phi)} \int_0^t \frac{q(s)}{(t-s)^{1-\Phi}} ds$$, t ∈ J

Now we are going to prove the following existence theorem for the FHDEs of system (1).

Theorem 2:
Let's assume that hypotheses (FH1)-(FH3) hold. Then the fractional hybrid differential equations of system (1) has a solution definedon J.
Proof:
Set Ṣ =C(J, □) and a subset Ũ of Ṣ defined by
Ũ={p∈Ṣ |‖p‖≤N},

Where, $N \ge L + Z_0 + \frac{T^{\Phi}}{\Gamma(\Phi+1)} \|\mathrm{F}\|_{L^1}$

Obviously Ũ is a closed, convex as well as bounded subset of Banach space Ṣ. Thus we follow the system (1).

It is clear that, p(t) is a solution of fractional hybrid differential equations system (1) if & only if p(t) satisfy the integral equations given below:

$$\begin{Bmatrix} p(t) = f(t, p(t)) + \frac{1}{\Gamma\Phi}\int_0^t \frac{g(s, q(s), I^{\Phi}(q(s)))}{(t-s)^{1-\Phi}} ds \\ q(t) = f(t, q(t)) + \frac{1}{\Gamma\Phi}\int_0^t \frac{g(s, p(s), I^{\Phi}(p(s)))}{(t-s)^{1-\Phi}} ds \end{Bmatrix}, \quad t \in J$$

(3)

Define two operators A: Ṩ → Ṩ and B: Ũ → Ṩ by

$$\begin{Bmatrix} Ap(t) = f(t, p(t)), \\ Bp(t) = \frac{1}{\Gamma(\Phi)}\int_0^t (t-s)^{\Phi-1} g(s, p(s), I^{\Phi}(p(s))) ds \end{Bmatrix}, t \in J$$

So the problem (3) can be transformed into the system of operator equation in such manner:

$$\begin{Bmatrix} p(t) = Ap(t) + Bq(t) \\ q(t) = Aq(t) + Bp(t) \end{Bmatrix}, t \in J$$

We are showing that these operators satisfy all the concepts of theorem 1.

P,q is element of □ by hypothesis (FH1), we have

$$|Ap(t) - Aq(t)| = |f(t, p(t)) - f(t, q(t))| \le \frac{L(|p(t)-q(t)|)}{2(M + |p(t)-q(t)|)} \le \frac{L(\|p-q\|)}{2(M + \|p-q\|)}, \text{ for all } t \in J$$

By taking the Chebyshev over t, we get

$$\|Ap - Aq\| \le \frac{L(\|p-q\|)}{2(M + \|p-q\|)},$$

This illustrate that A is a non-linear contraction on Ṩ with a control function Φ/2, where Φ is defined by:

$$\Phi(r) = \frac{Lr}{M + r}$$

Further, we demonstrate that B is constant as well as continuous operator on Ṩ.

Let's assume $\{p_n\}$ is a sequence in Ṩ converging to a point p ϵ Ṩ. Then

$$\lim_{n\to\infty} Bp_n(t) = \frac{1}{\Gamma(\Phi)} \lim_{n\to\infty}\left\{\int_0^t (t-s)^{\Phi-1} g(s, p(s), I^{\alpha}(p(s))) ds\right\}$$

$$= \frac{1}{\Gamma(\Phi)}\left\{\int_0^t (t-s)^{\Phi-1} \lim_{n\to\infty} g(s, p_n(s), I^{\alpha}(p_n(s))) ds\right\} = \frac{1}{\Gamma(\Phi)}\left\{\int_0^t (t-s)^{\Phi-1} g(s, p(s), I^{\alpha}(p(s))) ds\right\} = Bp(t)$$

for all t ϵ J, where the II^{nd} equality contains by Lebesgue dominated convergence theorem.

So B is a continuous function on Ṩ.

Let's assume p ∈Ṩ, by having (FH2), for all t∈J, we have

$$|Bp(t)| = \frac{1}{\Gamma(\Phi)}\left|\int_0^t (t-s)^{\Phi-1} g(s, p(s), I^{\alpha}(p(s))) ds\right|$$

$$\le \frac{1}{\Gamma(\Phi)}\int_0^t (t-s)^{\Phi-1} |g(s, p(s), I^{\alpha}(p(s)))| ds \le \frac{1}{\Gamma(\Phi)}\int_0^t (t-s)^{\Phi-1} F(s) ds \le \frac{T^{\Phi}}{\Gamma(\Phi+1)}\|F\|_{L^1}$$

Taking the chebyshev over t, we find $\|Bp\| \le \frac{T^{\Phi}}{\Gamma(\Phi+1)}\|F\|_{L^1}$

for all p ∈Ṩ , so it is clear that p is uniformly bounded over Ṩ .

Again let's assume t1, t2 ∈J, for any p ∈Ṩ one has

$$|Bp(t_1) - Bp(t_2)| = \frac{1}{\Gamma(\Phi)}\left|\int_0^{t_1}(t-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s))) - \int_0^{t_2}(t-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s)))\right|$$

$$\le \frac{1}{\Gamma(\Phi)}\left|\int_0^{t_1}(t_1-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s))) - \int_0^{t_2}(t_2-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s)))\right|$$

$$+\frac{1}{\Gamma(\Phi)}\left|\int_0^{t_1}(t_1-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s))) - \int_0^{t_2}(t_2-s)^{\Phi-1}g(s,p(s),D^{\alpha}(p(s)))\right|$$

$$\le \frac{\|F\|_{L^1}}{\Gamma(\Phi)}\left(\left|\int_0^{t_1}(t_1-s)^{\Phi-1} - (t_2-s)^{\Phi-1}ds\right| + \left|\int_{t_2}^{t_1}(t_2-s)^{\Phi-1}ds\right|\right)$$

$$\le \frac{\|F\|_{L^1}}{\Gamma(\Phi+1)}\left(|t_1^{\Phi} - t_2^{\Phi}| + |(t_2-t_1)^{\Phi}|\right)$$

Since $t^{\Phi}$is uniformly continuous on J for 0<Φ<1, for any ε>0 there exists δ1>0 such that if $|t_1 - t_2| < \delta 1$ we have

$$|t_1^{\Phi} - t_2^{\Phi}| < \frac{\Gamma(\Phi+1)}{2\|F\|_{L^1}}\varepsilon$$

Let's assume $\delta = \min\{\delta_1, (\frac{\Gamma(\Phi+1)}{2\|F\|_{L^1}}\varepsilon)^{\frac{1}{\Phi}}\}$ if $|t_2 - t_1| < \delta$, we have the following:

$$|Bp(t_1) - Bp(t_2)| < \frac{\|F\|_{L^1}}{\Gamma(\Phi+1)}\left(\frac{\Gamma(\Phi+1)}{2\|F\|_{L^1}}\varepsilon + \frac{\Gamma(\Phi+1)}{2\|F\|_{L^1}}\varepsilon\right) = \varepsilon$$

It illustrate that B(Ṩ) is equi-continuous. So it can be concluded that B is fully continuous over Ṩ .

To prove hypothesis (C) of theorem 1, let p ∈Ũ and q ∈Ṩ , such that p=Ap+Bq, by assumptions (FH1) and (FH2), we have

$$|p(t)| \le |Ap(t)| + |Bq(t)|$$

$$\le (|f(t,p(t)) - f(t,0)| + |f(t,0)|) + \frac{1}{\Gamma(\Phi)}\int_0^t (t-s)^{\Phi-1}|g(s,q(s),I^{\alpha}(q(s)))|ds$$

$$\le L + Z_0 + \frac{1}{\Gamma(\Phi)}\int_0^t (t-s)^{\Phi-1}F(s)ds$$

$$\le L + Z_0 + \frac{T^{\Phi}}{\Gamma(\Phi+1)}\|F\|_{L^1}$$

By taking chebyshev over t on J and by (FH2) we conclude this

$\|p\| \le L + Z_0 + \frac{T^{\Phi}}{\Gamma(\Phi+1)} \|F\|_{L^1} \le N$, which illustrate p ∈Ṣ .

So the assumption of theorem 1(c) has been proved. Moreover, all the constraints of theorem 1 are satisfied.

Hence, the operator T(p,q)=Ap+Bq has a coupled fixed point on Ũ.

Now, it is concluded as a results, that the fractional hybrid differential equations system (1) has a solution defined on J.

References

.

[1].J. H. Wu, “Theory and application of partial functional differential equations”, Springer-Verlag, Berlin-New York, 1996.

[2]. T. A. Burton, “Integral equations, implicit functions and fixed points”, Proc. Amer. Math. Soc., 124, pp. 2383-2390, 1996.

[3]. T. A. Burton and Colleen Kirk, “A fixed point theorem of Krasnoselskii-Schaefer type”, MathematischeNachrichten, 189, pp. 23-31, 1998.

[4]. H. Schaefer, “Uber die Methode der a priori-Schranken”, Math. Ann. 129, pp. 415-416, 1955.

[5]. C. S. Barroso, “Krasnoselskii’s fixed point theorem for weakly continuous maps”, Nonlinear Anal., 55, pp. 25-31, 2003.

[6]. Khan, JA, Zahoor, RMA, Qureshi, “Swarm intelligence for the problem of non-linear ordinary differential equations and its application to well knownWessinger’s equation”,Europian Journal Sci. Res.,Vol. 4, pp. 514-525, 2009.

[7]. Dhage, BC, “Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations”, Differ. Equ. Appl., Vol. 2, pp. 465-486, 2010.

[8]. Dhage, BC, Salunkhe, SN, Agarwal, RP, Zhang, W, “A functional differential equation in Banach algebras”, Math. Inequal. Appl., Vol. 8, pp. 89-99, 2005.

[9]. Afshari, H, Kalantari, S, Karapinar, E, “Solution of fractional differential equations via coupled fixed point”, Electronic Journal of Differential Equation, Vol. 286, 2015.

[10]. Dhage, BC, Jadhav, NS, “Basic results in the theory of hybrid differential equations with linear perturbations of second type”,Tamkang J. Math., Vol.44, pp. 171-186, 2013.

[11]. Diethelm, K, Ford, NJ,“Multi-order fractional differential equations and their numerical solution” Applied Mathematical Computation, Vol. 154, pp. 621-640, 2004.

[12]. Wang, G, Zhang, L, Song, G,“Boundary value problem of a nonlinear Langevin equation with two different fractional orders and impulses”, Fixed Point Theory Appl., Vol. 200, 2012.

[13]. Jleli, M, Karapinar, E, Samet, B,“Positive solutions for multipoint boundary value problems for singular fractional differential equations”, J. Appl. Math., Article ID 596123, 2014.

[14]. Caraballo, T, Diop, MA, Ndiaye, AA, “Asymptotic behavior of neutral stochastic partial functional integro-differential equations driven by a fractional Brownian motion”, J. Nonlinear Sci. Appl., Vol. **7**, pp. 407-421, 2014.

[15]. Al-sawalha, MM, Shoaib, M,“Reduced-order synchronization of fractional order chaotic systems with fully unknown parameters using modified adaptive control”, J. Nonlinear Sci. Appl., Vol. 9, pp. 1815-1825, 2016.
[16]. Lu, H, Sun, S, Yang, D, Teng, H,“Theory of fractional hybrid differential equations with linear perturbations of second type”, Boundary Value Problem, Vol. 23, 2013.
[17]. Gejji, VD,“Positive solutions of a system of non-autonomous fractional differential equations”, J. Math. Anal. Appl. 302, pp. 56-64, 2005.
[18]. Chen, Y, An, H, “Numerical solutions of coupled Burgers equations with time and space fractional derivatives”, Applied Mathematics Computation, Vol. 200, pp. 87-95, 2008.
[19]. Ahmad, B, Alsaedi, A,“Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations”, Fixed Point Theory Appl., Article ID 364560, 2010.
[20]. Su, X, “Boundary value problem for a coupled system of nonlinear fractional differential equations”, Applied Mathematics Letter, Vol. 22, pp. 64-69, 2009.
[21]. Gafiychuk, V, Datsko, B, Meleshko, V, Blackmore, D, “Analysis of the solutions of coupled nonlinear fractional reaction-diffusion equations”, Chaos Solitons Fractals 41, pp. 1095-1104 2009.
[22]. Kilbas, AA, Srivastava, HM, Trujillo, JJ, “Theory and Applications of Fractional Differential Equations”, Elsevier, Amsterdam, 2006.
[23]. Podlubny, I, “Fractional Differential Equations”,Academic Press, New York, 1999.
[24]. Chang, SS, Cho, YJ, Huang, NJ, “Coupled fixed point theorems with applications”, Journal of Korean Mathematics Society, Vol. 33, pp. 575-585, 1996.