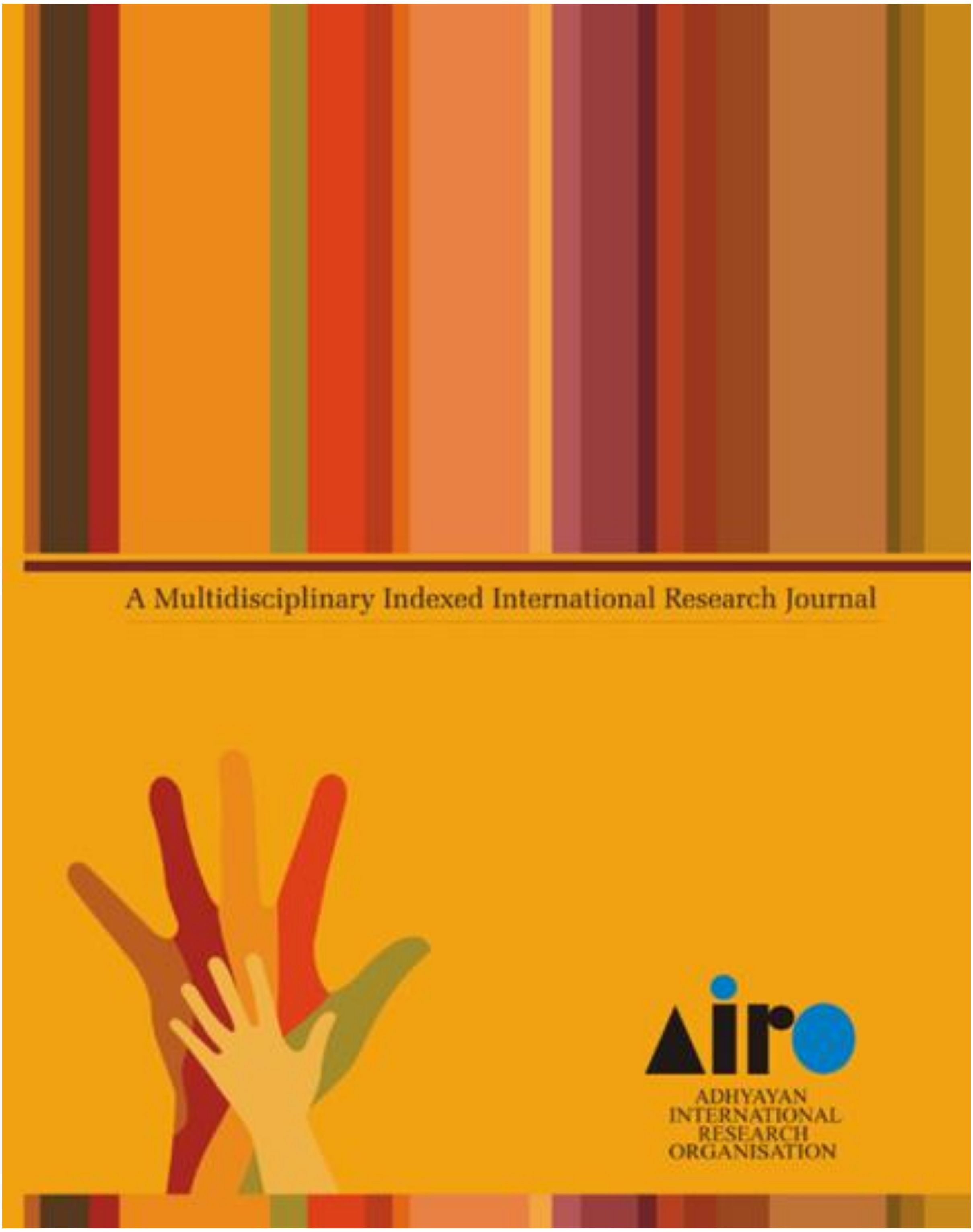
Airo International Research Journal

Volume XII, ISSN: 2320-3714
July, 2017
Impact Factor 0.75 to 3.19

ज्ञान-विज्ञान विमुक्तये
UGC Approval Number 63012



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EXAMINATION OF VARIOUS STUDIES IN SEMI INNER PRODUCT SPACES

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ABSTRACT

In mathematics, the semi-inner-product is a generalization of inner products formulated by Günter Lumer for the purpose of extending Hilbert space type arguments to Banach spaces in functional analysis. Fundamental properties were later explored by Giles. The "semi-inner product" in standard functional analysis textbooks,where a "semi-inner product" satisfies all the properties of inner products (including conjugate symmetry) except that it is not required to be strictly positive.

KEYWORDS: mathematics, semi-inner-product, functional analysis.

INTRODUCTION

In the theory of operators on a Hubert space, the latter actually does not function as a particular Banach space (whose norm satisfies the parallelogram law), but rather as an inner-product space. It is in terms of the inner product space structure that most of the terminology and techniques are developed. On the other hand, this type of Hubert space considerations finds no real parallel in the general Banach space setting. Some time ago, while trying to carry over a Hubert space argument to a general Banach space situation, we were led to use a suitable mapping from a Banach space into its dual in order to make up for the lack of an inner product. Our procedure suggested the existence of a general theory which it seemed should be useful in the study of operator (normed) algebras by providing better insight on known facts, a more adequate language to "classify" special types of operators, as well as new techniques. These ideas evolved into a theory of semi-inner-product spaces which is presented in this paper (together with certain applications)('). We shall consider vector spaces on which instead of a bilinear form there is defined a form [x, y] which is linear in one component only, strictly positive, and satisfies a Schwarz inequality.

Such a form induces a norm, by setting ||x|| = ([x, x])1/2; and for every normed space one can construct at least one such form (and, in general, infinitely many) consistent with the norm in the sense [x, x] = ||x||2. In such a setting, one can then, for instance, talk about a pseudo quadratic form (we shall use the term "numerical range") of an operator T, i.e., [Tx, x]; one can define hermitian operators as those for which [Tx, x] is real; and one can extend the concept of a point state w to the case of an arbitrary algebra of normed space operators, by defining co(T) = [Tx, x], with x fixed. The important fact is that, roughly speaking, a semi-inner-product still provides one with sufficient structure to obtain certain nontrivial general results.

SEMI-INNER PRODUCTS

A semi-inner product on a complex linear space X is a mapping from X × X to C, denoted hx, yi for (x, y) ∈ X × X, where for all x, y, z ∈ X and for all α ∈ C we have:

(a) Linearity in the first variable:

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$\langle \alpha x, z\rangle = \alpha\langle x, z\rangle.$$

(b) Conjugate symmetry: $\langle y, x\rangle = \overline{\langle x, y\rangle}.$

(c) Positivity: $\langle x, x\rangle \geq 0$ (implying that $\langle x, x\rangle \in \mathbb{R}$).

If, in addition, $\langle x, x\rangle = 0$ implies x = 0, then $\langle \cdot, \cdot\rangle$ is an inner product. A linear space with a (semi) inner product is a (semi) inner product space.

Note: Property (a) can be summarized as

$$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

This combines with (b) to give "conjugate linearity" in the second variable:

$$\langle z, \alpha x + \beta y\rangle = \overline{\alpha}\langle z, x\rangle + \overline{\beta}\langle z, y\rangle.$$

Definition. For h·, ·i a semi-inner product, define the semi-norm

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Note. We easily have

$$\|\alpha x\| = \sqrt{\langle \alpha x, \alpha x\rangle} = \sqrt{\alpha\overline{\alpha}\langle x, x\rangle} = |\alpha|\|x\|.$$

Also, if $\langle \cdot, \cdot\rangle$ is an inner product, then $\|x\| = 0$ if and only if x = 0.

Example. For X = C^n , an inner product is

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)\rangle = \sum_{k=1}^{n} x_k \overline{y_k}.$$

Not surprisingly, for $f, g \in L^2$, an inner product on L^2 is given by $\langle f, g\rangle = \int_\Omega f\overline{g}$, as we will see.

Basic Identity. Let X be a semi-inner product space. Then the semi-norm induced by the semiinner product satisfies: for all x, y ∈ X, we have

Lemma 1.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\mathrm{Re}\langle x, y\rangle.$$

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Theorem 1. Cauchy-Schwartz Inequality:

Given a semi-inner product on X, for all x, y ∈ X we have $|\langle x, y\rangle| \leq \|x\| \|y\|$ if $\langle\cdot,\cdot\rangle$ is an inner product, then equality holds if and only if x is a scalar multiple of y (that is, x and y are linearly independent).

Note. As you've seen before, the Cauchy-Schwartz Inequality is used to prove the Triangle Inequality.

Theorem 2.Triangle Inequality. Let X be a semi-inner product space. Then the semi-norm induced by the semiinner product satisfies: for all x, y ∈ X, we have $\|x + y\| \leq \|x\| + \|y\|$.

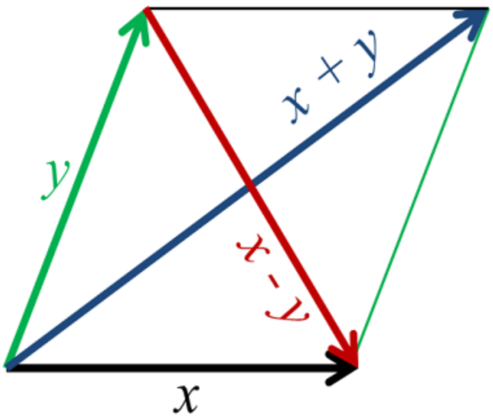
Note. With the Triangle Inequality established, we see that if $\langle\cdot,\cdot\rangle$ is a (semi) inner product, then $\|\cdot\|$ is a (semi) norm.

Note. We have seen many normed linear spaces. In the case of $\mathbb{R}^n$, $\mathbb{C}^n$, L^2, and ℓ^2, the (standard Euclidean) norms are "induced" by an inner product. However, these are special cases. The following result can be used to show that, among the L^p spaces, only for p = 2 is the norm induced by an inner product.

Proposition 1.Parallelogram Law. Let X be a semi-inner product space. Then the semi-norm induced by the semi inner product satisfies: for all x, y ∈ X, we have

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Note. The reason this is called the Parallelogram law is that, in a parallelogram, the sum of the squares of the diagonals equals the sum of the squares of the lengths of the four edges:



Notice that the contrapositive of the Parallelogram law can be read as: "If the semi-norm in a normed linear space does not satisfy the Parallelogram Law, then the semi-norm is not induced by an inner product." This can be used to show that the ` p spaces are not inner product spaces for 1 ≤ p < 2 and 2 < p ≤ ∞.

Note. We know that if $\langle\cdot,\cdot\rangle$ is an inner product, then $\|x\| = \sqrt{\langle x, x\rangle}$ is a norm. The following result gives the inner product in terms of the norm induced by it.

Proposition.Polarization Identity. Let X be a semi-inner product space. Then the semi-norm

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induced by the semi inner product satisfies: for all x, y ∈ X, we have

$$\langle x, y\rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right)$$

Note. The following result tells us when a norm is induced by an inner product

Theorem 3. If $\|\cdot\|$ is a norm on a (complex) linear space X satisfying the parallelogram law, then the norm is induced by an inner product.

Proof. Homework. See the text for hints

Theorem 4. Continuity of Inner Product

In any semi-inner product space, if the sequence $(x_n) \to x$ and $(y_n) \to y$, then $(\langle x_n, y_n\rangle) \to \langle x, y\rangle$

Definition. A Hilbert space is a complete inner product space

NORMALIZED DUALITY MAPPINGS

The normalized duality mapping of a normed space is defined as follows. X∗ denotes the dual space of a normed space X, and P(X∗), the power set of X∗.

Definition 1. Let $(X, \|\cdot\|)$ be a normed space. The mapping $\mathcal{J} : X \to \mathcal{P}(X^*)$ given by

$$\mathcal{J}(x) := \{f \in X^* : f(x) = \|x\|^2 \text{ and } \|f\| = \|x\|\}$$

is called the normalized duality mapping of X. The next result contains some fundamental properties of the set valued mapping J [15]. We include the proof here, since it is an interesting application of the Hahn-Banach extension theorem.

Theorem 1. Let $(X, \|\cdot\|)$ be a normed space. Then the following statements are true:

(a) *For each $x \in X$, the set $\mathcal{J}(x)$ is a nonempty convex subset of X^*;*
(b) *For all $x \in X$ and every $\lambda \in \mathbb{K}$, $\mathcal{J}(\lambda x) = \overline{\lambda}\mathcal{J}(x)$.*

Proof. (a) Let x ∈ X. By the very definition of the normalized duality mapping (Definition 1.3.1), J (x) is a subset of X∗ .

If $x=0$, then clearly $\mathcal{J}(0) = \{0\}$. If $x \neq 0$, consider the subspace S_x:= span{x} of X. Define the functional $g : S_x \to \mathbb{K}$ by $g(u) = \lambda\|x\|^2$ where $u = \lambda x \in S_x$ $(\lambda \in \mathbb{K})$. Then for all $u = \lambda x,\ v = \mu x \in S_x$ $(\lambda, \mu \in \mathbb{K})$, and for all $\alpha,\ \beta \in \mathbb{K}$, we have

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$$\begin{aligned} g(\alpha u+\beta v) &= g((\alpha\lambda+\beta\mu)x) \\ &= (\alpha\lambda+\beta\mu)\|x\|^2 \\ &= \alpha\, g(u)+\beta\, g(v), \end{aligned}$$

so that g is linear on S_x. Further, for all $u=\lambda x\in S_x \ (\lambda\in\mathbb{K})$ we have

$$\begin{aligned} |g(u)| &= \left|\lambda\|x\|^2\right| \\ &= \|\lambda x\|\,\|x\| \\ &= \|x\|\,\|u\|, \end{aligned}$$

so that g is also bounded on S_x, and $\|g\|=\|x\|$. Therefore, by virtue of the HahnBanach extension theorem, there exists a functional $f\in X^*$ which extends g to the whole of X such that $\|f\|=\|g\|=\|x\|$. Also, since $x\in S_x$, we have

$$f(x)=g(x)=g(1.x)=\|x\|^2.$$

It follows that $f\in\mathcal{J}(x)$ so that $\mathcal{J}(x)$ is nonempty.

Now we will show that $\mathcal{J}(x)$ is convex. To this end, suppose that $x\neq 0$, and let $f_1,\ f_2\in\mathcal{J}(x)$. Then for every $\lambda\in[0,1]$, we have

$$\lambda f_1+(1-\lambda)\,f_2\in X^*,$$

And,

$$\begin{aligned} (\lambda f_1+(1-\lambda)f_2)(x) &= \lambda f_1(x)+(1-\lambda)\,f_2(x) \\ &= \lambda\|x\|^2+(1-\lambda)\|x\|^2 \\ &= \|x\|^2. \end{aligned}$$

Hence for every $\lambda\in[0,1]$, we have

$$\begin{aligned} 0<\|x\| &= \left|(\lambda f_1+(1-\lambda)f_2)\left(\frac{x}{\|x\|}\right)\right| \\ &\le \sup_{0\neq y\in X}\left|(\lambda f_1+(1-\lambda)\,f_2)\left(\frac{y}{\|y\|}\right)\right| \\ &= \|\lambda f_1+(1-\lambda)f_2\|, \end{aligned}$$

which shows that $\|x\| \leq \|\lambda f_1 + (1-\lambda) f_2\|$. However, since $f_1, f_2 \in \mathcal{J}(x)$, we have $\|f_1\| = \|f_2\| = \|x\|$ so that

$$\|\lambda f_1 + (1-\lambda) f_2\| \leq \lambda \|f_1\| + (1-\lambda) \|f_2\| = \|x\|$$

for every $\lambda \in [0,1]$. Hence for every $\lambda \in [0,1]$, $\|\lambda f_1 + (1-\lambda) f_2\| = \|x\|$, and this completes the proof of (a).

$$\frac{1}{\overline{\lambda}} f \in X^*,$$

$$\left(\frac{1}{\overline{\lambda}} f\right)(x) = \frac{1}{\overline{\lambda}} f(x) = \|x\|^2,$$

And,

$$\left\|\frac{1}{\overline{\lambda}} f\right\| = \frac{1}{|\overline{\lambda}|} \|f\| = \frac{1}{|\overline{\lambda}|} \|\lambda x\| = \|x\|.$$

This shows that

$$\frac{1}{\overline{\lambda}} f \in \mathcal{J}(x) \text{ so that } f \in \overline{\lambda} \mathcal{J}(x).$$

Thus we have the inclusion $\mathcal{J}(\lambda x) \subseteq \overline{\lambda} \mathcal{J}(x)$. A similar argument leads to the the reverse inclusion. Hence holds (b) also.

CONCLUSION

The concept of semi-inner product has made great progress. In the setting of a normed space, a semi-inner product provides a sufficient structure as well as new techniques for obtaining some nontrivial general results. It plays an important role in the theory of accretive operators and dissipative operators, differential equations, linear and nonlinear semi groups in Banach spaces, and Banach space geometry. To us, this concept is important from the view point of best approximation theory. We employ this concept in the characterization of best approximations in normed spaces. Our main concern is to derive some results characterizing best approximations in the framework of a general normed space through a semi-inner product that generates the norm of the space.

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