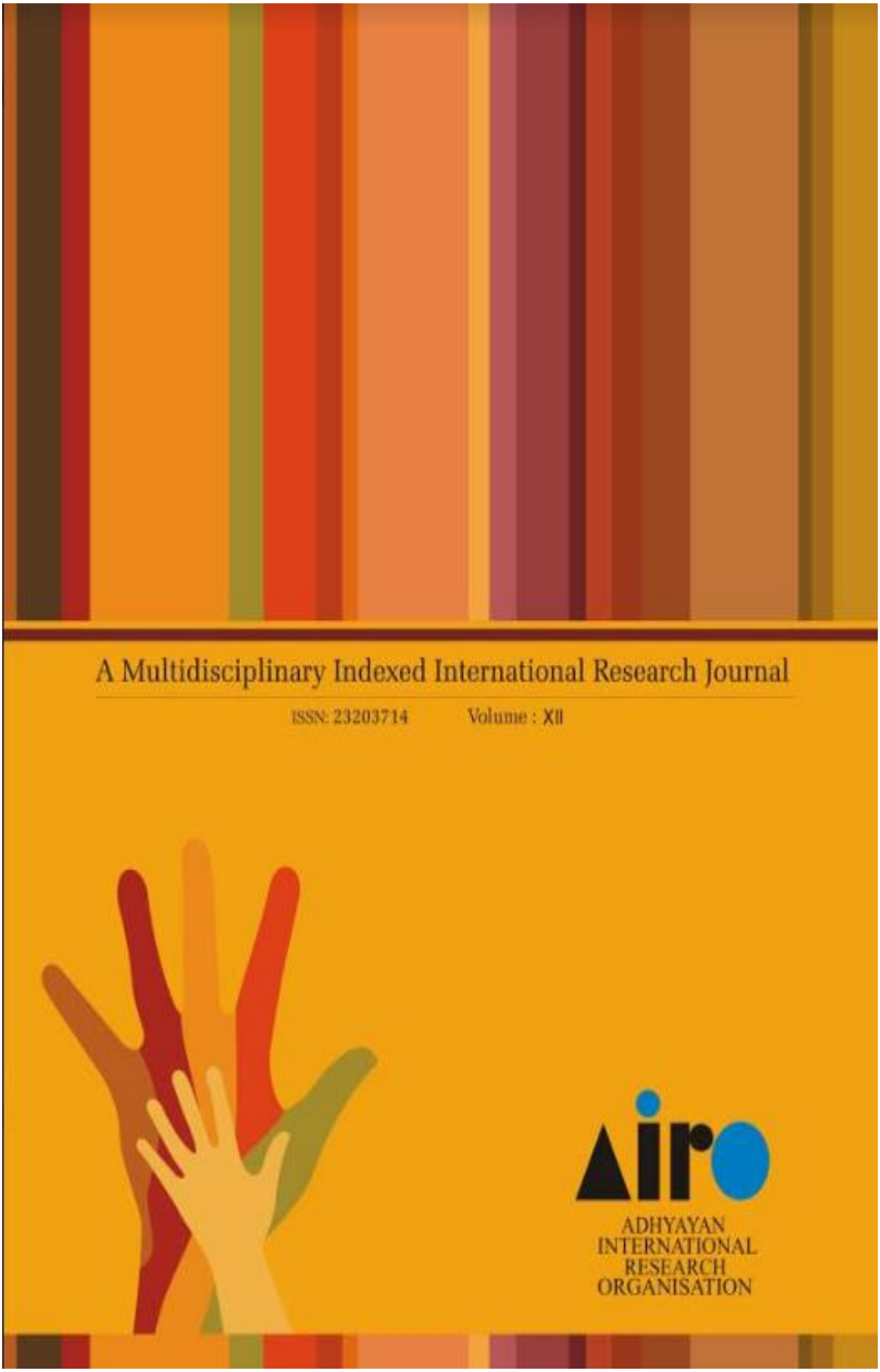
A Multidisciplinary Indexed International Research Journal
ISSN: 23203714 Volume : XII
Airo
ADHYAYAN
INTERNATIONAL
RESEARCH
ORGANISATION

A STUDY ON SIGNIFICANCE OF IDEMPOTENT MATRICES' AND ITS APPLICATIONS

Farooq Ahamad Kuchay

Research Scholar Jodhpur National University

ABSTRACT

In matrix theory, we come across many special types of matrices and one among them is idempotent matrix, which plays an important role in functional analysis especially spectral theory of transformations and projections. Idempotent matrices are closely associated with the theory of generalized inverses. Benjamin Peirce was an American mathematician who introduced the term 'idempotent' first ever in 1870. The term 'idempotent' describes a mathematical quantity which remains unchanged when multiplied by itself. A complex matrix that satisfies is known as idempotent matrix.

KEYWORDS: *Idempotent, matrices, mathematical*

INTRODUCTION

Generalizing the concept of idempotent matrices via permutations, a special type of matrix namely k-idempotent matrix is introduced and studied in this thesis as a continuation of -real, hermitian and k-EP matrices in literature. The algebraic, geometrical and topological nature of real numbers was revealed only when it had been generalized into an arbitrary field. Generalization of concrete concepts into an abstract nature is always done in mathematics just for the sake of developing the subject and also to determine the source of characterizations availed by such mathematical concepts.

With the exception of the identity matrix, an idempotent matrix is singular; that is, its number of independent rows (and columns) is less than its number of rows (and columns). This can be seen from writing $MM = M$, assuming that M has full rank (is non-singular), and pre-multiplying by M^{-1} to obtain $M = M^{-1}M = I$.

Theorem

Let A and B be two k-idempotent matrices. If B is k-similar to A then B^3 is similar to A^3.

Proof

Since B is k-similar to A, we can find a non-singular matrix P such that

$$B = KP^{-1}KAP$$

$$KB = P^{-1}KAP$$

$$(KB)^2 = P^{-1}KAPP^{-1}KAP$$

$$(KB)^2 = P^{-1}(KA)^2P$$

$$B^3 = P^{-1}A^3P$$

Hence B^3 is similar to A^3. ∎

When an idempotent matrix is subtracted from the identity matrix, the result is also idempotent. This holds since $[I - M][I - M] = I - M - M + M^2 = I - M - M + M = I - M$.

A matrix A is idempotent if and only if for any natural number n, $A^n = A$. The 'if' direction trivially follows by taking n=2. The `only if' part can be shown using proof by induction. Clearly we have the result for n=1, as $A^1 = A$. Suppose that $A^{k-1} = A$. Then, $A^k = A^{k-1}A = AA = A$, as required. Hence by the principle of induction, the result follows.

An idempotent matrix is always diagonalizable and its eigen values are either 0 or 1. The trace of an idempotent matrix — the sum of the elements on its main diagonal — equals the rank of the matrix and thus is always an integer. This provides an easy way of computing the rank, or alternatively an easy way of determining the trace of a matrix whose elements are not specifically known (which is helpful in econometrics, for example, in establishing the degree of bias in using a sample variance as an estimate of a population variance).

Literature Review

A.R. Meenakshi and S. Krishnamoorthy introduced the concept of range hermitian (or k-EP) as a generalization of hermitian matrices. A theory for k-EP matrix is developed which reduces to that of EP matrices as a special case, when 'k' is the identity permutation. Characterizations of a k-EP matrix analogous to that of an EP matrix are determined. Relations between k-EP and EP matrices are discussed. Necessary and sufficient conditions are derived for a matrix to be k-EP. The conditions for the sums and products of k-EP matrices to be k-EP are investigated. Various generalized inverses and in particular the group inverse of a k-EP matrix to be EP is analyzed. Necessary and sufficient conditions for the product of k-EP partitioned matrices to be k-EP and for Schur complement of in a partitioned matrix of the form to be - are obtained. Necessary and sufficient conditions for a k-EP matrix to have its principal sub matrices and their Schur complement to be k-EP are determined.

R. E. Hartwig and M. S. Putcha discovered set of conditions under which a matrix T could be written as a sum of idempotent matrices as well as a difference of two idempotent matrices.

It was shown by Pei Yuan Wu that any complex matrix is a sum of finitely many idempotent matrices if and only if tr T is an integer and tr T >=rank T.

The characterization of product of idempotent matrices was studied by J. Hannah and K. C. O'Meara. The minimum number of idempotents needed in such a product is determined thereby generalizing the result of C.S. Ballantine.

A result of J.A Erdos states that if is a singular matrix with entries in field then can be written as the product of idempotents.

T. J. Laffey considered the case, where is replaced by a ring. He showed that if R is a division ring or Euclidean ring then every singular n x n matrix with entries in can be expressed as a product of idempotent over R.

J. Benitez and N. Thome discussed the idempotency of linear combination of an idempotent matrix and a -potent matrix that commute. This paper deals with idempotent matrices and -potent matrices when both matrices commute.

M. Sarduvan and H. Ozdemir discussed the problem of linear combination of two tripotent, idempotent and involutive matrices to be idempotent.

Example

Examples of a 2×2 and a 3×3 idempotent matrix are $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, respectively.

Real 2 × 2 case

If a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is idempotent, then

- $a = a^2 + bc$
- $b = ab + bd$ implies $b(1 - a - d) = 0$ so $b = 0$ or $d = 1 - a$.
- $d = bc + d^2$

If *b* = *c*, the matrix $\begin{pmatrix} a & b \\ b & 1-a \end{pmatrix}$ will be idempotent provided $a^2 + b^2 = a,$ so *a* satisfies the quadratic equation

$a^2 - a + b^2 = 0,$ or $(a - \frac{1}{2})^2 + b^2 = \frac{1}{4}$

which is a circle with center (1/2, 0) and radius 1/2. In terms of an angle θ,

$M = \frac{1}{2}\begin{pmatrix} 1 - \cos\theta & \sin\theta \\ \sin\theta & 1 + \cos\theta \end{pmatrix}$ is idempotent.

However, *b* = *c* is not a necessary condition: any matrix

$\begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$ with $a^2 + bc = a$ is idempotent.

APPLICATIONS OF IDEMPOTENT MATRIX

Idempotent matrices arise frequently in regression analysis and econometrics. For example, in ordinary least squares, the regression problem is to choose a vector β of coefficient estimates so as to minimize the sum of squared residuals (mis-predictions) e_i: in matrix form,

$$\text{Minimize } (y - X\beta)^T(y - X\beta)$$

where y is a vector of dependent variable observations, and X is a matrix each of whose columns is a column of observations on one of the independent variables. The resulting estimator is

$$\beta = (X^TX)^{-1}X^Ty$$

where superscript T indicates a transpose, and the vector of residuals is

$$e = y - X\beta = y - X(X^TX)^{-1}X^Ty = [I - X(X^TX)^{-1}X^T]y = My.$$

Here both M and $X(X^TX)^{-1}X^T$ (the latter being known as the hat matrix) are idempotent and symmetric matrices, a fact which allows simplification when the sum of squared residuals is computed:

$$e^Te = (My)^T(My) = y^TM^TMy = y^TMMy = y^TMy.$$

The idempotency of M plays a role in other calculations as well, such as in determining the variance of the estimator β.

Theorem

Let A be a k-idempotent matrix. Then $I - A$ is k-idempotent if and only if A is idempotent.

Proof

$$I - A = K(I - A)^2K$$

$$= K(I - 2A + A^2)K$$

$$= I - 2A^2 + A$$

Hence $2(A - A^2) = 0$,which implies that A is idempotent.

Conversely, if A is idempotent then A commutes with the permutation matrix K (cf. *lemma 2.2.5*)

$$K(I - A)^2K = K(I - 2A + A^2)K$$

$$= K(I - A)K$$

$$= I - A$$

Hence $I - A$ is k-idempotent. ■

Theorem

Let A and B be k-idempotent matrices. If $AB = BA$ then AB is also a k-idempotent matrix.

Proof

$$AB = KA^2KKB^2K$$

$$= KA^2B^2K$$

$$= KAABBK$$

$$= K(AB)^2K \qquad \text{[by } AB = BA\text{]}$$

Hence the matrix AB is k-idempotent. ■

An idempotent linear operator *P* is a projection operator on the range space *R(P)* along its null space *N(P)*. *P* is an orthogonal projection operator if and only if it is idempotent and symmetric.

The concept of k-idempotent matrices is introduced for complex matrices and exhibited as a generalization of idempotent matrices. It is shown that k-idempotent matrices are quadripotent. The conditions for power hermitian matrices to be k-idempotent are obtained. It is also proved that the set $G=\{A,A^2,A^3,KA,AK,KA^3\}$ forms a group under matrix multiplication. It is shown that a -idempotent matrix reduces to an idempotent matrix if and only if AK=KA.

Theorem

Let P and Q be a pair of idempotent matrices of order m , and let a_1 *and* a_2 *be two nonzero scalars such that* $a_1+a_2\neq 0$.

$r(a_1P+a_2Q)=r(P+Q),$

that is , the rank of a_1P+a_2Q *is invariant with respect to* $a_1\neq 0, a_2\neq 0$ *and* $a_1+a_2\neq 0$.

Theorem.

Let P and Q be a pair of idempotent matrices of order m.

$$r(P+Q)=r\begin{bmatrix} P & Q & 0 \\ Q & 0 & P \end{bmatrix}-r[P,Q],$$

$$r(P+Q)=r\begin{bmatrix} P & Q \\ Q & 0 \\ 0 & P \end{bmatrix}-r\begin{bmatrix} P \\ Q \end{bmatrix},$$

$$r(P+Q)=r\begin{bmatrix} P-QP \\ Q-PQ \end{bmatrix}+r(P)+r(Q)-r[P,Q],$$

$$r(P+Q)=r[P-PQ,Q-QP]+r(P)+r(Q)-r\begin{bmatrix} P \\ Q \end{bmatrix}.$$

Proof.

$$r\begin{bmatrix} P & Q & 0 \\ Q & 0 & P \end{bmatrix} = r\begin{bmatrix} P & Q-PQ & 0 \\ Q-QP & -Q & P \end{bmatrix}$$

$$= r(P) + r\begin{bmatrix} 0 & Q-PQ & 0 \\ Q-QP & -Q & P \end{bmatrix}$$

$$= r(P) + r\begin{bmatrix} 0 & Q-PQ & 0 \\ Q-QP & -PQ & P \end{bmatrix}$$

$$= r(P) + r\begin{bmatrix} 0 & Q-PQ & 0 \\ Q-QP & 0 & P \end{bmatrix}$$

$$= r(P) + r(Q-PQ) + r[Q-QP, P]$$

$$= [P, Q] + r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} - r(P)$$

$$= r[P, Q] + r(P+Q) \quad (\textbf{by } (2.1)).$$

RESEARCH METHODOLOGY

Let P and Q be a pair of complex idempotent matrices of order m. Then the sum P+Q satisfies the following rank equalities.

$$r(P+Q) = r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} - r(Q) = r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix} - r(P),$$

r(P+Q)=r[P-PQ,Q]=r[Q-QP,P],

$$r(P+Q) = r\begin{bmatrix} P-QP \\ Q \end{bmatrix} = r\begin{bmatrix} Q-PQ \\ P \end{bmatrix},$$

r(P+Q)=r(P-PQ-QP+QPQ)+r(Q),

r(P+Q)=r(Q-PQ-QP+PQP)+r(P).

Proof

Recall that the rank of a matrix is an important invariant quantity under elementary matrix operations for this matrix, that is, these operations do not change the rank of the matrix. Thus, we first find by elementary block matrix operations the following trivial result:

$$r\begin{bmatrix} P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -P-Q \end{bmatrix} = r(P) + r(Q) + r(P+Q).$$

On the other hand, note $P^2=P$ and $Q^2=Q$. By elementary block matrix operations, we also obtain another nontrivial rank equality

$$r\begin{bmatrix} P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} P & 0 & P \\ -QP & 0 & Q \\ P & Q & 0 \end{bmatrix} = r\begin{bmatrix} 2P & 0 & P \\ 0 & 0 & Q \\ P & Q & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 2P & 0 & 0 \\ 0 & 0 & Q \\ 0 & Q & \frac{1}{2}P \end{bmatrix}$$
$$= r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix} + r(P).$$

Combining the above two results yields the first equality. By symmetry, we have the second equality. A matrix X is called a generalized inverse of A if $AXA=A$, and is denoted as $X=A^-$. Clearly, any idempotent matrix is a generalized inverse of itself. Applying the following rank equalities due to Marsaglia and Styan

$r[A,B]=r(A)+r(B-AA^-B)=r(B)+r(A-BB^-A)$,

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^-A) = r(C) + r(A - AC^-C),$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r[(I_m - BB^-)A(I_n - C^-C)],$$

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r\begin{bmatrix} A & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix}$$
$$= r(A) + r\begin{bmatrix} 0 & B - AA^-B \\ C - CA^-A & D - CA^-B \end{bmatrix}$$

SIGNIFICANCE OF THE STUDY

The present research work can be very helpful for future prospective. In this study, work is done on eigen and k-eigen values of an idempotent matrix and methods are studied to find these values, which can be helpful. Also, various generalized inverses of an idempotent matrix are determined and problems are discussed related to idempotent matrix.

REFERENCES

1. Ann Lee, ”Secondary symmetric, secondary skew symmetric, secondary orthogonal matrices”, Period. Math. Hungary, 7 , (2006) : 63 76.

2. Baksalary J.K, “Relationship between the star and minus orderings”, Lin. Alg. Appl., 82, (2006): 163 167.

3. Baksalary. J.K and Baksalary. O.M, “Idempotency of linear combinations of two idempotent matrices”, Lin. Alg. Appl., 321, (2010) : 3 7.

4. Baksalary. J.K and Baksalary. O.M, “Commutativity of projectors”, Lin. Alg. Appl., 341, (2012): 129 142.

5. Baksalary. J.K and Baksalary. O.M and Tomasz Szule, “A property of orthogonal projectors”, Lin. Alg. Appl., 354, (2012): 35 39.

6. Baksalary. J.K and Baksalary. O.M and Styan. G.P.H, “Idempotency of linear combinations of an idempotent and a tripotent matrix”, Lin. Alg. Appl., 354, (2012): 21 34.

7. Baksalary. J.K, Pukelsheim. F and Styan. G.P.H, “Some properties of matrix partial orderings”, Lin. Alg. Appl., vol.119, (2009): 7 85.

8. Baksalary. O.M, “Idempotency of linear combinations of three idempotent matrices, two of which are disjoint”, Lin. Alg. Appl., 388, (2004) : 67 78.

9. Ballantine. C.S, “Products of idempotent matrices”, Lin. Alg. Appl., 19, (2008): 81 86.

10. Ben Israel .A and Greville. T.N.E, “Generalized inverses, Theory and applications”, Wiley and sons, New York, 2004.

11. Ben Israel. A and Greville. T.N.E, “Generalized inverses, Theory and applications” Springer- verlag, New York, 2003.

12. Benitez. J, “Idempotency of linear combinations of an idempotent and a t-potent matrix that do not commute”, Linear and multi linear algebra, vol.56, (2008): 679 687.