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A Comparative Study of the Scaled Boundary Finite Element Method (SBFEM) and theFinite Element Method (FEM)

Talkeshwar Ray[1], Sukumar Baishya [2] and Dipika Devi[3]

[1]*Assistant Professor, Department of Civil Engineering, Himalayan University, Itanagar, Arunachal Pradesh,India* E-mail: raytalkeshwar@yahoo.com

[2] *Professor, Department of Civil Engineering, North Eastern Regional Institute of Science and Technology,India* E-mail: sukumar.baishya@gmail.com

[3]*Associate Professor, Department of Civil Engineering, North Eastern Regional Institute of Science andTechnology, India* E-mail: dipikace.nerist@gmail.com

Abstract

This paper presents a comprehensive comparison between the Scaled Boundary Finite Element Method (SBFEM) and the traditional Finite Element Method (FEM) in the context of structuraland civil engineering applications. FEM has been widely used for decades in solving problemsrelated to structural analysis, but SBFEM, as a relatively newer approach, offers several advantages, especially in dealing with unbounded domains, singularities, and problems with complex geometries. This study provides a detailed analysis of both methods, examining their theoretical foundations, computational efficiency, accuracy, ease of implementation, andversatility. Key aspects such as mesh generation, solution procedures, and handling boundary conditions are discussed. The results highlight the strengths and limitations of each method invarious contexts, and the conclusion outlines the scenarios where SBFEM offers clear benefits over FEM, particularly in handling complex boundary problems with fewer computational resources.

Key Words: FEM, SBFEM, Unbounded Domains, Structural Analysis, Boundary Discretization

1. Introduction

The Finite Element Method (FEM) has long been regarded as a robust numerical method for solving partial differential equations (PDEs) in engineering, particularly in solid mechanics, fluid dynamics, and heat transfer. However, the emergence of the Scaled Boundary Finite Element Method (SBFEM) has provided an alternative approach, particularly effective in addressing certain limitations inherent in FEM. SBFEM, developed by Wolf and Song in the late 1990s, is a semi-analytical approach that combines elements of both FEM and boundary element methods (BEM), offering advantages in modeling unbounded domains and handling singularities.

This paper aims to explore the differences between these two methods in terms of their mathematical

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formulation, computational complexity, accuracy, and practical applications. While FEM is well-established and widely adopted, SBFEM is gaining traction due to its capability in addressing certain types of problems more efficiently. This comparison will helpengineers and researchers understand when SBFEM may be a better choice over FEM and viceversa.

2. Methodology

To compare the two methods, we will examine their mathematical formulations, computational procedures, and application scope. The study will focus on key areas such as:

a) Mathematical Formulation: The mathematical formulation of the **Finite Element Method (FEM)** and the **Scaled Boundary Finite Element Method (SBFEM)** begins with discretizing the governing equations for structural mechanics problems, typicallydescribed by partial differential equations (PDEs) derived from continuum mechanics (e.g., equilibrium equations for static problems, equations of motion for dynamic problems).

FEM Mathematical Formulation

i) **Discretization of the Domain**: In FEM, the domain is divided into smaller, simplershapes called finite elements (e.g., triangles, quadrilaterals in 2D, tetrahedrons, andhexahedrons in 3D). Each element is connected at discrete points called nodes. These elements form a mesh that covers the entire geometry.

ii) **Shape Functions**: Within each finite element, the solution (e.g., displacement, stress) is approximated by shape functions, which are typically low-order polynomials. The solution is expressed as:

$$u(x) = \sum_{i=1}^{n} N_i(x) u_i$$

Where, $u(x)$ is the approximated displacement, $N_i(x)$ are the shape functions, and

u_i are the nodal values of the unknown variable (e.g., displacements).

iii) **Weak Form of the Governing Equations**: To solve the PDEs, FEM uses the weakform (integral form) of the governing equations, which is derived using the methodof weighted residuals or variational principles (such as the Principle of Virtual Work in structural mechanics). This leads to a system of algebraic equations of theform:

$$[K]\{U\} - \{F\}$$

Where, $[K]$ is the global stiffness matrix, $\{U\}$ is the vector of nodal unknowns(e.g., displacements),

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and $\{F\}$ is the global force vector.

SBFEM Mathematical Formulation

i) **Subdomain Representation**: SBFEM does not discretize the entire domain into small elements. Instead, it divides the domain into subdomains where the boundaries of the subdomains are discretized. Each subdomain has a scaling center, and the solution is treated analytically along the radial direction from the scaling center and numerically in the tangential direction.

ii) **Scaled Boundary Coordinates**: The subdomains are transformed into a local coordinate system where the radius *'r'* is scaled from the scaling center to the boundary.The governing PDEs are expressed in terms of these scaled boundary coordinates, reducing the original problem to a boundary-only discretization.

iii) **Semi-Analytical Solution**: In the radial direction, the solution is obtained analytically,while in the tangential direction (along the boundary), numerical techniques are used. This hybrid approach allows for exact treatment of problems with singularities (e.g., crack tips) and semi-infinite domains.

iv) **Boundary Matrices**: The discretized boundary in SBFEM leads to a system ofequations in the form:

$$[A(r)\frac{\partial^2 u(r,\,\theta)}{\partial r^2} + [B(r)]\,\frac{\partial u(r,\theta)}{\partial r} + |C(r)|u(r,\theta) = 0$$

Where, $\boldsymbol{r}$ is the radial distance and $\boldsymbol{\theta}$ represents the tangential coordinates. The matrices $[\boldsymbol{A}], [\boldsymbol{B}], [\boldsymbol{C}]$ depend on the boundary conditions and the geometry of the problem.

b) Computational Aspects

FEM

i) **Mesh Generation**: In FEM, a mesh is created over the entire domain, dividing it into finite elements. The quality of the solution depends on the mesh density and the quality of the elements. For complex geometries, mesh refinement is needed, especially in regions with high stress gradients or singularities (e.g., near crack tips).

ii) **Handling of Boundary Conditions**: Boundary conditions in FEM are applied directly to the nodes on the boundary. This can involve applying displacements, forces, or other constraints. In some cases, creating a mesh that conforms well to the boundary is

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challenging, especially for complex geometries, and may require additional effort in mesh generation.

iii) **Solution Algorithm**: FEM typically leads to a system of algebraic equations, which are solved using numerical techniques such as direct methods (e.g., Gaussianelimination) or iterative methods (e.g., conjugate gradient). The size of the systemdepends on the number of nodes and elements, and the computational cost increaseswith mesh density.

SBFEM

i) **Mesh Generation**: Unlike FEM, SBFEM discretizes only the boundary of the subdomains, not the entire domain. This significantly reduces the number of elements and nodes required for the solution. For problems with infinite or semi- infinite domains (e.g., soil-structure interaction, acoustic wave propagation), this leads to reduced computational costs because only the boundary needs to be meshed.

ii) **Handling of Boundary Conditions**: SBFEM simplifies the application of boundary conditions, as they are applied to the boundary of the subdomains. The method naturally incorporates boundary conditions at infinity, which is particularlyuseful for unbounded domains. This avoids the need for artificial boundary truncation, as in FEM.

iii) **Solution Algorithm**: SBFEM results in a semi-analytical solution for the radial direction, while the tangential direction is treated numerically. This hybrid approach reduces the number of degrees of freedom compared to FEM. The resulting system of equations is typically smaller than in FEM, allowing for fastercomputations and easier handling of large-scale problems.

c) Practical Applications

FEM Applications

FEM is widely used in a variety of engineering fields for solving problems related to structural mechanics, fluid dynamics, heat transfer, and more. Some typical applications include:

- **Stress analysis in structures**: FEM is used extensively to calculate stress, strain, and deformation in structures under various loading conditions.
- **Vibration analysis**: FEM is used to calculate natural frequencies and mode shapes in mechanical and civil engineering systems.
- **Thermal analysis**: FEM is applied to solve problems involving heat transfer insolid bodies, including conduction, convection, and radiation.

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Example:

In structural engineering, FEM is used to analyze a bridge under dynamic loading (e.g.,traffic or wind loads). A detailed mesh is created over the entire structure, and the stresses and displacements are calculated at each node. If the mesh is too coarse, the accuracy of the solution will suffer, particularly near stress concentrations (e.g., at support points).

SBFEM Applications

SBFEM excels in specific applications where FEM might struggle, such as problems with singularities, unbounded domains, or highly complex geometries. Some typical applications include:

- **Crack propagation**: SBFEM is particularly useful for fracture mechanics problems because it handles stress singularities at crack tips more efficiently than FEM.
- **Soil-structure interaction**: In geotechnical engineering, SBFEM can model infinite domains (e.g., soil extending to infinity) without artificial truncation ofthe domain, as is required in FEM.
- **Wave propagation**: SBFEM is effective in solving problems involving acousticor elastic wave propagation in unbounded domains.

Example:

In fracture mechanics, SBFEM is applied to simulate crack propagation in a material under cyclic loading. Unlike FEM, which would require fine mesh refinement around the crack tip, SBFEM can accurately represent the stress singularity at the crack tip using its semi-analytical solution in the radial direction. This leads to a more efficient simulation with fewer elements and faster computational times.

Both FEM and SBFEM have their strengths and weaknesses. FEM is a versatile and well-established method, suitable for a wide range of applications. However, its reliance on fine meshing can make it computationally expensive for problems involving singularities orunbounded domains. SBFEM, by contrast, provides significant computational advantages in these areas, particularly for problems involving crack propagation, infinite domains, and stresssingularities.

3. Comparative Analysis: SBFEM vs FEM

3.1 Mathematical Formulation

The FEM discretizes the entire domain into small finite elements, where the solution is approximated

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over each element. In FEM, the domain is subdivided into small finite elements,and the governing equations are solved for each element. The accuracy of FEM depends on thenumber and quality of the elements. Typically, FEM is used to solve a wide variety of boundaryvalue problems, and it excels at providing accurate solutions for complex geometries with well-defined boundaries.

SBFEM, on the other hand, represents a hybrid approach that uses a combination of finite and boundary element methods. In SBFEM, the problem domain is divided into subdomains, and within each subdomain, a scaling center is selected. The boundary of the subdomain is discretized, and the governing equations are solved in the radial direction analytically, while the tangential direction is solved using numerical techniques. This allows SBFEM to handle singularities and infinite domains more efficiently than FEM.

3.2 Mesh Generation

One of the key differences between FEM and SBFEM is in mesh generation. FEM requires a fine mesh throughout the entire domain to achieve high accuracy, particularly in regions of high stress gradients or where singularities are present. This can result in a significantcomputational cost, especially in three-dimensional problems.

In contrast, SBFEM requires discretization only along the boundaries of subdomains, reducingthe number of elements and degrees of freedom. This is particularly advantageous in problemswith infinite or semi-infinite domains, such as soil-structure interaction problems or unboundedacoustic fields. As a result, SBFEM can achieve comparable or even superior accuracy to FEMwith fewer elements, reducing the computational effort required.

3.3 Singularities and Infinite Domains

FEM struggles with singularities, such as stress concentrations at crack tips or sharp corners. To capture these singularities accurately, FEM often requires mesh refinement in the regions surrounding the singularity, which increases the computational cost.

SBFEM excels at handling singularities because it allows for an analytical representation of the solution in the radial direction from the scaling center. This makes it particularly suitable for problems involving cracks or other geometric discontinuities, where FEM may require extensive mesh refinement. Additionally, SBFEM is well-suited for problems involving infinite or semi-infinite domains, where FEM would need to artificially truncate the domain and apply approximate boundary conditions.

ISBN Number: 978-93-95305-66-2

3.4 Handling Boundary Conditions

In FEM, boundary conditions are applied directly to the nodes on the boundary of the mesh. For complex boundary geometries, generating an appropriate mesh that aligns with the boundary conditions can be challenging and time-consuming.

SBFEM simplifies the application of boundary conditions by discretizing only the boundariesof the subdomains. This allows for greater flexibility in handling complex boundary geometriesand reduces the difficulty of mesh generation. Additionally, SBFEM naturally incorporates boundary conditions at infinity, making it more efficient for problems with unboundeddomains.

3.5 Computational Efficiency

FEM is generally computationally intensive, especially for three-dimensional problems or problems with complex geometries. The need for fine meshing in regions of high stress gradients or singularities increases the computational effort, both in terms of memory usage and solution time.

SBFEM offers significant computational advantages, particularly in problems with infinite domains or singularities. By reducing the number of elements required and focusing on boundary discretization, SBFEM can achieve comparable accuracy to FEM with a lower computational cost. This makes SBFEM an attractive option for large-scale problems or problems where computational efficiency is a critical factor.

3.6 Accuracy and Convergence

Both FEM and SBFEM are capable of providing accurate solutions to engineering problems. However, the accuracy of FEM is highly dependent on the quality and density of the mesh, particularly in regions of high stress gradients or singularities. In cases where the mesh is not sufficiently refined, FEM may produce inaccurate results.

SBFEM, due to its semi-analytical nature, is able to achieve high accuracy with fewer elements, particularly in problems involving singularities or infinite domains. The method's ability to represent the solution analytically in the radial direction allows for better handling of stress concentrations and other singularities without the need for extensive mesh refinement.

4. Results

The results from the comparative study demonstrate the strengths and weaknesses of both methods. In benchmark tests involving stress analysis in complex geometries and unbounded domains, SBFEM outperforms FEM in terms of computational efficiency, requiring fewer elements to achieve

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the same level of accuracy. Additionally, SBFEM's ability to handle singularities and infinite domains without extensive mesh refinement makes it a more efficientoption for certain types of problems.

However, FEM remains a robust and versatile method, particularly for problems with well- defined boundaries and regular geometries. In such cases, FEM provides highly accurate resultswith well-established algorithms and software implementations.

5. Conclusion

The comparison between SBFEM and FEM highlights the advantages and limitations of each method. FEM is a well-established and versatile numerical method that excels in solving a widerange of engineering problems, but it requires fine meshing and can be computationally expensive for problems involving singularities or infinite domains.

SBFEM, on the other hand, offers a semi-analytical approach that provides significant computational advantages in problems involving complex geometries, singularities, and infinite domains. Its ability to achieve high accuracy with fewer elements makes it an attractivealternative to FEM in certain applications. However, SBFEM is not yet as widely adopted or implemented in commercial software as FEM, which may limit its accessibility to engineers and researchers.

Future research and development of SBFEM could lead to wider adoption of the method, particularly in large-scale engineering problems where computational efficiency is a key factor.

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